7.1 – Orthogonal Matrices

Definition: A square matrix A is said to be **orthogonal** if its transpose is the same as its inverse, that is, if $A^{-1} = A^T$ or, equivalently, if $AA^T = A^TA = I$. A matrix transformation $T_A : R^n \to R^n$ is said to be an **orthogonal transformation** or an **orthogonal operator** if A is an orthogonal matrix.
Consider the matrix in Exercise 6.

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

Theorem 7.1.1 The following are equivalent for an $n \times n$ matrix A.
a) A is orthogonal.
b) The row vectors of A form an orthonormal set in R^n with the Euclidean inner product.
c) The column vectors of A form an orthonormal set in R^n with the Euclidean inner product.

Theorem 7.1.2
a) The transpose of an orthogonal matrix is orthogonal.
b) The inverse of an orthogonal matrix is orthogonal.
c) A product of orthogonal matrices is orthogonal.
d) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

#8 Let $T_A : R^3 \to R^3$ be multiplication by the orthogonal matrix in Exercise 6. Find $T_A(\mathbf{x})$ for the vector $\mathbf{x} = (0,\ 1,\ 4)$, and confirm $\|T_A(\mathbf{x})\| = \|\mathbf{x}\|$ relative to the Euclidean inner product on R^3.

$$A = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}$$

Theorem 7.1.3 If A is an $n \times n$ matrix, then the following are equivalent.

a) A is orthogonal.

b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x}$ in R^n [length is preserved].

c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}$ and $\mathbf{y}$ in R^n [the dot product is preserved].

Example 4: Rotations in the xy-plane

#14 Consider the rectangular $x'y'$-coordinate system obtained by rotating a rectangular xy-coordinate system counterclockwise through the angle $\theta = 3\pi/4$.
a. Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2,\ 6)$.
b. Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5,\ 2)$.

Theorem 7.1.5 Let V be a finite-dimensional inner product space. If P is the transition matrix from one orthonormal basis for V to another orthonormal basis for V, then P is an orthogonal matrix.

Theorem 7.1.4 If S is an orthonormal basis for an n-dimensional inner product space V, and if $(\mathbf{u})_S = (u_1, u_2, \ldots,\ u_n)$ and $(\mathbf{v})_S = (v_1, v_2, \ldots,\ v_n)$ then:
a) $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$
b) $d\,(\mathbf{u},\ \mathbf{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)^2}$
c) $\langle \mathbf{u},\ \mathbf{v} \rangle = u_1v_1 + u_2v_2 + \ldots + u_nv_n$

#**23** The set $S = \left\{ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}x, \sqrt{\frac{3}{2}}x^2 - \sqrt{\frac{2}{3}} \right\}$ is an orthonormal basis for P_2 with respect to the evaluation inner product at the points $x_0 = -1$, $x_1 = 0$, $x_2 = 1$. Let $\mathbf{p} = p(x) = 1 + x + x^2$ and $\mathbf{q} = q(x) = 2x - x^2$.

a. Find $(\mathbf{p})_S$ and $(\mathbf{q})_S$.

b. Use Theorem 7.1.4 to compute $\|\mathbf{p}\|$, $d(\mathbf{p}, \mathbf{q})$, and $\langle \mathbf{p}, \mathbf{q} \rangle$.